Elementary Numerical Simulation of a Single-Phase Reservoir

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INTRODUCTION

This paper contains an expository discussion of the basic difference equations used in the approximate solution of the differential equations describing isothermal laminar flow of fluids in a reservoir. The discussion will be focused on a particular example in which the calculations will be fully illustrated. The interested reader can find more complete discussions in the literature. Although Professor Douglas' survey of methods[1] is recommended, the bibliography in this author's book[2] may be used for additional guidance.

The Darcy flow of a single phase in a horizontal (gravity is neglected) linear reservoir will be described by the partial differential equation

$$\phi \frac{\partial \rho}{\partial t} = \frac{\partial}{\partial x}\left(\frac{k\rho}{\mu}\frac{\partial p}{\partial x}\right) \tag{1}$$

in which the English-American field unit system[3] will be used as shown in Table 1.

TABLE 1

Symbol	Name	Units
k	permeability	perms (1 darcy=6.329 perm)
p	pressure	pounds per square inch
t	time	days
x	distance	feet
μ	viscosity	centipoise
ρ	density	pounds per cubic foot
ϕ	porosity	dimensionless

Equation 1 will be discussed in relation to a hypothetical one-dimensional reservoir which is 100 feet long. Suppose the porosity to be a constant 0.20 and suppose the permeability to be 10 millidarcies (= 0.010 darcies). Suppose the pores of the reservoir to be filled with a homogeneous fluid having a viscosity of 0.3164 centipoise (roughly water at 95°C). Suppose the density of this fluid is described by

$$\rho = 60.0 \exp\left[0.01\,(p - 1000.0)\right]. \tag{2}$$

The above specifications of viscosity and density are not intended to precisely represent any liquid. That is, these specifications are designed to give definite values for illustration instead of genuine representation of a specific fluid.

Discussion of the above reservoir and the fluid it contains will begin with the solution of a simplified reservoir flow problem in order that the results can be used for comparison with approximate results obtained by finite difference methods. Several methods will be compared to a limited extent with an intuitive feeling about these difference methods as a goal.

CLOSED FORM SOLUTION

The slightly compressible fluid described by (2) requires that pressure should be eliminated from (1).* Since (2) implies that

$$\frac{d\rho}{dp} = 0.01\,\rho, \tag{3}$$

and the chain rule implies that

$$\frac{\partial \rho}{\partial x} = \frac{\partial p}{\partial x}\frac{d\rho}{dp} = 0.01\,\rho\,\frac{\partial p}{\partial x}; \tag{4}$$

(1) can be written as

$$0.01\,\phi\frac{\partial \rho}{\partial t} = \frac{\partial}{\partial x}\left(\frac{k}{\mu}\frac{\partial \rho}{\partial x}\right) \tag{5}$$

**Numbers in parentheses refer to appropriate equation.*

Next, substitute $\phi = 0.20$ and

$$k/\mu = (6.329)\,(0.01)\,/\,(0.3164) = 2.00 \times 10^{-1} \qquad (6)$$

to write (5) as

$$0.002\frac{\partial \rho}{\partial t} = 0.200\,\frac{\partial^2 \rho}{\partial x^2}\,. \qquad (7)$$

That is,

$$\frac{\partial \rho}{\partial t} = (100.0)\frac{\partial^2 \rho}{\partial x^2}\,. \qquad (8)$$

Density could have been eliminated from (1) to produce an equation showing pressure as a dependent variable. Density is a more convenient dependent variable in this case.

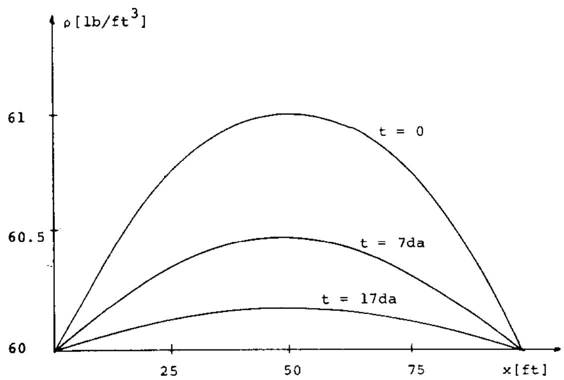

FIGURE 1

THREE SOLUTIONS OF PROBLEM

Since every function of the form

$$\rho\,(x, t) = [A \cos(0.1\lambda x) + B \sin(0.1\lambda x)]\; e^{-\lambda^2 t} \qquad (9)$$

is a solution of (8) where A, B, and λ are arbitrary real constants, a particular solution must be specified by a choice of initial and boundary conditions. Specifically, suppose that density is initially given by

$$\rho\,(x, 0) = 60.0 + \sin(\pi x/100), \qquad (10)$$

where $x = 0.0$ represents one end of the reservoir while $x = 100.0$ represents the other end. Adopt the boundary conditions given by

$$\rho\,(0, t) = \rho\,(100, t) = 60.0,\;\; t > 0. \qquad (11)$$

It can be shown that the initial condition in (10) and the boundary conditions in (11) imply that

$$\rho\,(x, t) = 60 + \sin(\pi x/100)\exp(-\pi^2 t/100) \qquad (12)$$

is the correct unique solution of (8).

Figure 1 shows the density in terms of position for several time levels. The vertical scale is exaggerated for purposes of display. Observe that the variation in density tends to smooth out as a function of time. Specifically, density is constant at infinite time.

Pressure tends to smooth out as a function of time as a consequence of the equation of state in (2). Since (12) implies that

$$\frac{\partial \rho}{\partial x} = (\pi/100)\cos(\pi x/100)\exp(-\pi^2 t/100), \qquad (13)$$

then

$$\frac{\partial p}{\partial x} = (\pi/\rho)\cos(\pi x/100)\exp(-\pi^2 t/100). \qquad (14)$$

Define the flux v by the equation

$$v = -\frac{k}{\mu}\frac{\partial p}{\partial x} \qquad (15)$$

and use (6) in (14) to obtain

$$v\,(0, t) = (-\pi/300)\exp(-\pi^2 t/100), \qquad (16)$$

and

$$v(100, t) = (\pi/300)\exp(-\pi^2 t/100). \qquad (17)$$

The smoothing of pressure as time increases implies a decrease in the total mass of fluid in the reservoir. Equations (16) and (17) describe this production. For example, (17) states that the initial ($t = 0$) fluid flow rate at the right ($x = 100$) end of the reservoir is equal to ($\pi/300$) cubic feet per day through each square foot of outlet face.

Although the foregoing discussion of production is not essential to the comparisons to be studied later, it is useful in gaining understanding of the mechanisms in the smoothing of pressure and density in time.

Of course, the problem discussed above suffers from the flaw inherent in examples in that a workable example is likely to be so simple that the reader is misled by its simplicity. Nevertheless, it forms the basis for the approximate solution methods that follow.

EXPLICIT METHOD

Choose five equally spaced points in the reservoir as illustrated in Fig. 2. Although a larger number of these points could have been chosen for greater accuracy, five will suffice for this illustration. These five points will be called *nodes*. Each node has an associated block as schematically illustrated by rectangles in the figure. The cross-section for flow is perpendicular to the plane of the figure. The appropriate cross-sectional area is irrelevant here unless a conversion of production to barrels is desired. Indeed, the vertical thickness of the reservoir is equally irrelevant even though it is shown in the figure.

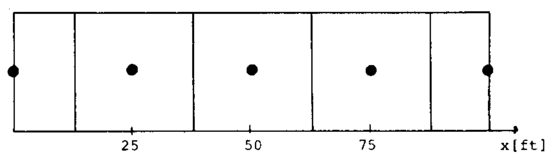

FIGURE 2

NODAL ARRANGEMENT

The x coordinates of the nodes are denoted by $x_0, x_1, \ldots, x_{n+1}$. The value of n is three in this case; and this illustrates the fact that the formulas below will be written with sufficient generality that they could be applied to other examples if desired. Similarly, let Δx denote the spacing between nodes (Δx = 25 ft in the example).

The main idea of each finite difference method is the estimation of the dependent variable (ρ in this case) at each node and at each time in a list of times $t_0, t_1, t_2, \ldots$. It is conventional to set t_0 to zero, and it is convenient to suppose $t_j = j\Delta t$ for j = 1, 2, ... For example, $t_1 = \Delta t$ will be 2.5 days in these particular illustrations.

Of course, $\rho(x_i, t_j)$ is known by means of (12) for this example. In any case, let u_{ij} denote an estimate of $\rho(x_i, t_j)$. The explicit method is given by the equation

$$u_{i,j+1} - u_{i,j} = \theta(u_{i-1,j} - 2u_{i,j} + u_{i+1,j}), \tag{18}$$

where θ is a constant defined by

$$\theta = 100.0\,\Delta t/(\Delta x)^2 = 0.40 \tag{19}$$

when Δt = 2.5 days and Δx = 25 feet.

The explicit method can be described by the statement that the space derivative is approximated at time t_j while the time derivative is approximated by a forward difference. Observe that (18) refers to three space subscripts (i-1, i, i+1) paired with the time subscript j while only one term uses the time subscript (j+1). That is, for each i, (18) contains only one unknown if it is supposed u_{ij} is known for every value of the space subscript i. Figure 3 illustrates this configuration.

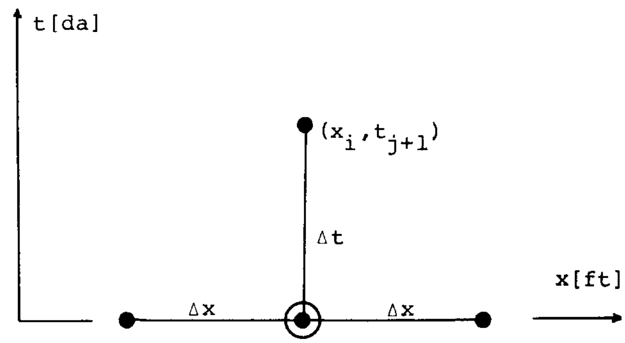

FIGURE 3

EXPLICIT METHOD

The reader's first impression may well be that (18) is so simple that it should be universally applied, and no other methods will be needed. Unfortunately, nothing could be further from the facts. That is, this deceptively simple scheme has a severe limitation. Specifically, this scheme fails unless Δt is quite small due to a phenomenon known as instability. A detailed discussion of stability is beyond the scope of this paper; one must be content with the remark that Δt=2.5 da is small enough.

Since is given at the ends of the reservoir, $u_{0,j}$ and $u_{n+1,j}$ are chosen to agree with this data. Also, $u_{i,0}$ is chosen in agreement with the initial conditions. Thus, set i = 1 in (18) to obtain

$$u_{1,j+1} = \theta u_{0,j} + (1-2\theta)u_{1,j} + \theta u_{2,j}. \tag{20}$$

Similarly,

$$u_{2,j+1} = \theta u_{1,j} + (1-2\theta)u_{2,j} + \theta u_{3,j}, \tag{21}$$

and

$$u_{3,j+1} = \theta u_{2,j} + (1-2\theta)u_{3,j} + \theta u_{4,j}. \tag{22}$$

Write (20) through (22) in matrix form as

$$\begin{pmatrix} u_{1,j+1} \\ u_{2,j+1} \\ u_{3,j+1} \end{pmatrix} = \begin{pmatrix} 1-2\theta & \theta & 0 \\ \theta & 1-2\theta & \theta \\ 0 & \theta & 1-2\theta \end{pmatrix} \begin{pmatrix} u_{1,j} \\ u_{2,j} \\ u_{3,j} \end{pmatrix} + \begin{pmatrix} \theta u_{0,j} \\ 0 \\ \theta u_{4,j} \end{pmatrix} \tag{23}$$

which is abbreviated as

$$u^{(j+1)} - Au^{(j)} + b^{(j)} . \tag{24}$$

That is, the list of values of u at the later time $t_{j+1} = t_j + \Delta t$ is given by addition of a boundary condition vector $b^{(j)}$ to the product of a propagation matrix A and the vector $u^{(j)}$ listing values of u at the earler time t_j. Observe that

$$A = \begin{pmatrix} 0.2 & 0.4 & 0.0 \\ 0.4 & 0.2 & 0.4 \\ 0.0 & 0.4 & 0.2 \end{pmatrix} \text{ and } b^{(j)} = \begin{pmatrix} 24.0 \\ 0 \\ 24.0 \end{pmatrix} \tag{25}$$

in this special case. Equations (24) and (25) will be useful in later sections for comparison.

The explicit or forward difference method consists of repeated application of (20) through (22) until some desired number of time steps have been taken. The results of two time steps are shown in Table 2. The first number in each little box is the correct density as given by (12) while the second number is the approximation given by use of (20) through (22) or by (23).

TABLE 2

t \ x	25.0	50.0	75.0
0.0	60.707 60.707	61.000 61.000	60.707 60.707
2.5	60.552 60.541	60.781 60.766	60.552 60.541
5.0	60.432 60.415	60.610 60.585	60.432 60.415

The two numbers in each little box in Table 2 differ for several reasons. Specifically, some *round-off* appears (the lower right number should perhaps be 60.414 or 60.416); the *truncation error* (error due to use of finite differences for derivatives) has some effect; *propagated error* (error in earlier steps causing error in the current step) may have some effect; and, finally, the author may have blundered in the calculations.

If no blunder appears in Table 2, then the major source of error is the truncation error. The values in the table may seem to be in very close agreement at first glance. However, since it is known that each answer is very close to 60.0, the purpose of the calculation is to determine deviation from this value. Table 3 presents the results in terms of those deviations.

TABLE 3

t \ x	25.0	50.0	75.0
0.0	.707 .707	1.000 1.000	.707 .707
2.5	.552 .541	.781 .766	.552 .541
5.0	.432 .415	.610 .585	.432 .415

Table 3 does not seem to indicate the computations to be as accurate as those indicated in Table 2. If not satisfied, redo the work with Δx reduced, say with x = 12.5 ft. However, if Δx is halved, Δt will have to be one-fourth as large as before for stability[4]. That is, one would need to use (0,0.625,1.25,1.875,2.5,. . ., 5.0) as a list of times in calculation of answers for comparison with those in Table 2 and Table 3. Observe that the number of unknowns at each step would be 7 = 8-1. Although the calculation is omitted here, the above discussion gives some notion of the work the calculation would involve. The concept of stability implies that propagated error does not play a disastrous role relative to local truncation error if the restriction on Δt is followed. Distaste for such restrictions is a prime motivation for the use of implicit methods.

IMPLICIT METHOD

The implicit method is described by the statement that the space derivative is approximated at time t_{j+1} while the time derivative is approximated by a backward difference. Specifically, modify (18) to obtain

$$u_{i,j+1} - u_{i,j} = \theta(u_{i-1,j+1} - 2u_{i,j+1} + u_{i+1,j+1}), \quad (26)$$

where θ is given by (19) just as before. Write (26) in the form

$$-\theta u_{i-1,j+1} + (1+2\theta)u_{i,j+1} - \theta u_{i+1,j+1} = u_{i,j}. \quad (27)$$

Observe that (27) contains three unknowns

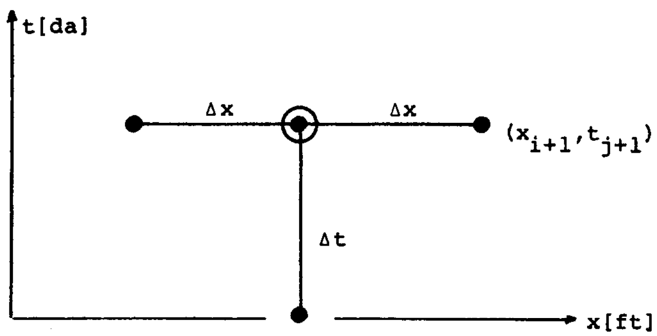

FIGURE 4

IMPLICIT METHOD

while (18) contains only one unknown. A comparison of Fig. 3 and Fig. 4 illustrates this distinction between the two methods.

The example (see Fig. 2) takes the form

$$\begin{pmatrix} 1+2\theta & -\theta & 0 \\ -\theta & 1+2\theta & -\theta \\ 0 & -\theta & 1+2\theta \end{pmatrix} \begin{pmatrix} u_{1,j+1} \\ u_{2,j+1} \\ u_{3,j+1} \end{pmatrix} = \begin{pmatrix} u_{1,j} \\ u_{2,j} \\ u_{3,j} \end{pmatrix} + \begin{pmatrix} 60\,\theta \\ 0\,\theta \\ 60\,\theta \end{pmatrix}. \quad (28)$$

Now consider (28) with θ again equal to 0.4 based on a Δt of 2.5 days. The first step in the construction of a table similar to Table 2 involves the solution of the matrix equation

$$\begin{pmatrix} 1.8 & -0.4 & 0 \\ -0.4 & 1.8 & -0.4 \\ 0 & -0.4 & 1.8 \end{pmatrix} \begin{pmatrix} u_{1,1} \\ u_{2,1} \\ u_{3,1} \end{pmatrix} = \begin{pmatrix} 60.707 \\ 61.000 \\ 60.707 \end{pmatrix} + \begin{pmatrix} 24.0 \\ 0 \\ 24.0 \end{pmatrix}. \quad (29)$$

That is,

$$\left.\begin{aligned} 1.8u_{1,1} - 0.4u_{2,1} \qquad\qquad &= 84.707 \\ -0.4u_{1,1} + 1.8u_{2,1} - 0.4u_{3,1} &= 61.000 \\ -0.4u_{2,1} + 1.8u_{3,1} &= 84.707 \end{aligned}\right\} \quad (30)$$

It is interesting to note that the notation

$$A = \begin{pmatrix} 1+2\theta & -\theta & 0 \\ -\theta & 1+2\theta & -\theta \\ 0 & -\theta & 1+2\theta \end{pmatrix}^{-1} \quad (31)$$

allows (28) to be written as

$$u^{(j+1)} = Au^{(j)} + A\begin{pmatrix} 60\,\theta \\ 0 \\ 60\,\theta \end{pmatrix} \quad (32)$$

which has the same abstract form as (24) if

$$b^{(j)} = A\begin{pmatrix} 60\,\theta \\ 0 \\ 60\,\theta \end{pmatrix}. \quad (33)$$

Now, evaluate the estimates. Observe that (30) is such that $u_{1,1}$ is clearly the same as $u_{3,1}$. That is, substituting $u_{1,1}$ for $u_{3,1}$ and vice versa simply puts the third equation first and vice versa without altering the set of equations. Thus, solve

$$\left.\begin{aligned} 1.8u_{1,1} - 0.4u_{2,1} &= 84.707 \\ -0.8u_{1,1} + 1.8u_{2,1} &= 61.000 \end{aligned}\right\}. \quad (34)$$

From (34), $u_{3,1} = u_{1,1} = 60.573$ and $u_{2,1} = 60.811$. These results are then inserted in (28) to accomplish a second time step. The results are summarized in Table 4 and Table 5 for comparison with Table 2 and Table 3, respectively.

CRANK-NICOLSON METHOD

A comparison of Table 3 and Table 5 leads to several observations. First, it appears that the explicit method worked rather better than the implicit method in this example. Although this is true in the example, it is *not* true in general. Indeed, the above two methods should give roughly the same accuracy (of course, stability requirements must be adhered to in the explicit case). On the other hand, observe

TABLE 4

t \ x	25.0	50.0	75.0
0.0	60.707	61.000	60.707
	60.707	61.000	60.707
2.5	60.552	60.781	60.552
	60.573	60.811	60.573
5.0	60.432	60.610	60.432
	60.464	60.655	60.464

TABLE 5

t \ x	25.0	50.0	75.0
0	.707	1.000	.707
	.707	1.000	.707
2.5	.552	.781	.552
	.573	.811	.573
5.0	.432	.610	.432
	.464	.655	.464

that the tables seem to show that the correct answers lie between the answers obtained by the above two methods. Moreover, the latter observation is valid in the general case.

The Crank-Nicolson method can be described as the "arithmetic mean" of the implicit method and the explicit method. That is, add (18) to (26) and divide by 2.0 to obtain

$$u_{i,j+1}-u_{i,j} = (u_{i-1,j+1}-2u_{i,j+1}+u_{i+1,j+1})\theta/2 + (u_{i-1,j}-2u_{i,j}+u_{i+1,j})\theta/2 \quad , \tag{35}$$

where θ is given by (19). Figure 5 illustrates the nodal configuration in this case. The results of (35) for this example are shown in Table 6 and Table 7.

The Crank-Nicolson method can be stated in the form given by (24) if A and $b^{(j)}$ are properly defined. Let A_E denote the matrix given in (23) and let A_I be based on (28) and defined by

$$A = \begin{pmatrix} 1+2\theta & -\theta & 0 \\ -\theta & 1+2\theta & -\theta \\ 0 & -\theta & 1+2\theta \end{pmatrix} \tag{36}$$

Write (23) and (28) respectively as

$$u^{(j+1)} = A_E u^{(j)} + \begin{pmatrix} 60\ \theta \\ 0 \\ 60\ \theta \end{pmatrix} \tag{37}$$

and

$$u^{(j+1)} = A_I^{-1} u^{(j)} + A_I^{-1} \begin{pmatrix} 60\ \theta \\ 0 \\ 60\ \theta \end{pmatrix} \tag{38}$$

Formally average these expressions to obtain

$$u^{(j+1)} = \tfrac{1}{2}(A_E + A_I^{-1}) + \tfrac{1}{2}(I + A_I^{-1}) \begin{pmatrix} 60\ \theta \\ 0 \\ 60\ \theta \end{pmatrix} \tag{39}$$

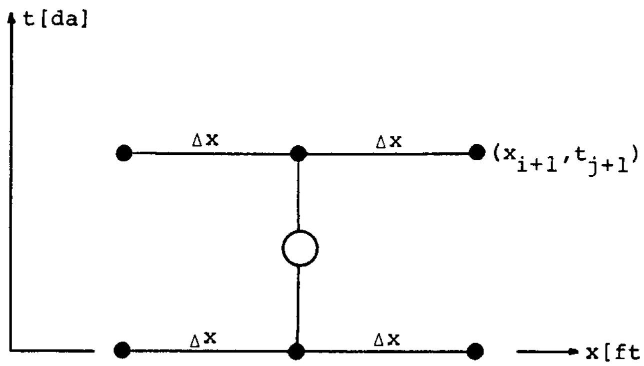

FIGURE 5

CRANK-NICOLSON METHOD

and (24) applies with the obvious identification of A and $b^{(j)}$. Observe that (39) could have been obtained directly from (35).

REMARKS

The above discussion is necessarily quite sketchy. It is hoped that it is of some benefit as an introduction.

It is unfortunate that the subject of stability

TABLE 6

t \ x	25.0	50.0	75.0
0.0	60.707 60.707	61.000 61.000	60.707 60.707
2.5	60.552 60.557	60.781 60.788	60.552 60.557
5.0	60.432 60.440	60.610 60.620	60.432 60.440

TABLE 7

t \ x	25.0	50.0	75.0
0.0	.707 .707	1.000 1.000	.707 .707
2.5	.552 .557	.781 .788	.552 .557
5.0	.432 .440	.610 .620	.432 .440

could not be discussed in more detail. Chapters 1 and 6 of this author's book[2] expand this example in some detail.

It should be noted that the propagation matrices of the respective finite difference methods determine the question of stability or instability in each case. Both the implicit and the Crank-Nicolson method are always stable. This means that they are more reliable than the explicit method in most cases.

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