Point-By-Point Calculation of the Time Variation of the Polished Rod Load in Sucker Rod Pumped Oil Wells

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ABSTRACT

Developed is an analytical analysis from which a synthesized dynamometer card can be calculated and plotted from generally known oil well parameters. The analysis preserves the time of displacement variable necessary for the calculation of instantaneous loads at any time or any position of the polished rod throughout the pumping cycle. The nonlinear boundary conditions introduced by the fluid pump are linearized and result in the applicability of superposition of loads in proper phase relationship.

From the synthesized dynamometer cards, various pumping conditions may be investigated; surface and subsurface equipment selected; and malfunction of system components determined.

ANALYTICAL CONSTRUCTION OF DYNAMOMETER CARDS

INTRODUCTION

For the calculation of the maximum polished rod load occurring during the pumping cycle numerous formulae and analytical procedures have been developed and are in use. Certain of these take into account the distributed nature of the rod string system; however, the non-linear boundary conditions arising from the intermittent operation of the pump valves have not been considered in analytical methods. It is the purpose of this paper to develop a method by which dyanamometer cards may be analytically constructed from generally known or calculable well parameters. The intermittent application of the fluid load is considered in its proper time phase relationship, and linearization techniques are applied to allow this fluid load to be transferred to the polished rod position. This fluid load is then added directly to the dynamic load arising from the motion of the sucker rod mass.

The dynamometer card, as recorded by a well dynamometer, is a force-displacement graph of the instantaneous load at the polished rod. The dynamometer card then is a record of the variation in load that is due to the movement of the rod string, fluid column, and surface equipment. Assuming constant input velocity or constant angular crank velocity, one finds that the load is the result of this velocity acting on the movable parts of the overall system. This paper is devoted only to an analysis of the rod line part of the over-all system; therefore, certain idealizations of surface equipment, bottom hole pump, and distributed fluid column are made. By comparison of the analytical card with dynamometer cards an acceptable method of calculation would give investigators a means of selecting surface and subsurface equipment, establishing pumping conditions, and determining malfunction of equipment.

OBJECTIVES

The primary objective of this paper is to develop a method of analyzing the rod line system. As a result of the assumed input velocity acting on the rods and of the consideration of the superimposed intermittent fluid column load in its proper time phase relationship, this analysis should produce a final expression of the polished rod or input force versus time. For the purpose of this analysis the distributed fluid column will be considered as a lumped or constant load acting on the rods over a part of the upstroke cycle. Justification for this consider action is found in the observation of pump dynamometer cards and from the analysis of the fluid column as a distributed system. Apparently at nominal pumping speeds the dynamic fluid load transferred to the rods is negligible in comparison with the static fluid load. A second objective is to develop the necessary analytical procedure by which the results -- its magnitude and phase relationship at the polished rod -- of the application of the intermittent static fluid load may be evaluated.

ANALYTICAL MODEL

As previously stated, this analysis is limited to one element, namely the rod line system, of the pumping system. Ignoring then the unit and its associated equipment, one finds it necessary to assume the nature of the input velocity. For the basic analysis it will be assumed that the input velocity is sinusoidal in nature and that the distortion -- arising from intermittent fluid load, mass of surface equipment counterbalance, etc. -- can be considered by Fourier Analysis.

Figure 1 schematically shows the rod line system, together with information relative to nomenclature, points of load and velocity application and waveforms. As indicated by the arbitrary element of the rod line, the mass, damping and elasticity are considered to be uniformly distributed along the entire length of the rod system, and each distributed element is assumed to possess uniform size, density, and surface friction or damping. Denoting the instantaneous force by f and instantaneous velocity by v, the equation of continuity and D'Alemberts Principal results in two equations. J. R. Norton (1) develops these in detail.

$$\frac{\partial f}{\partial x} = cv + M \frac{\partial v}{\partial t} \qquad 1.$$

$$\frac{\partial v}{\partial x} = 1/k \frac{\partial f}{\partial t} \qquad 2.$$

These equations express the relationship between force, velocity, mass, damping and the elastic properties of

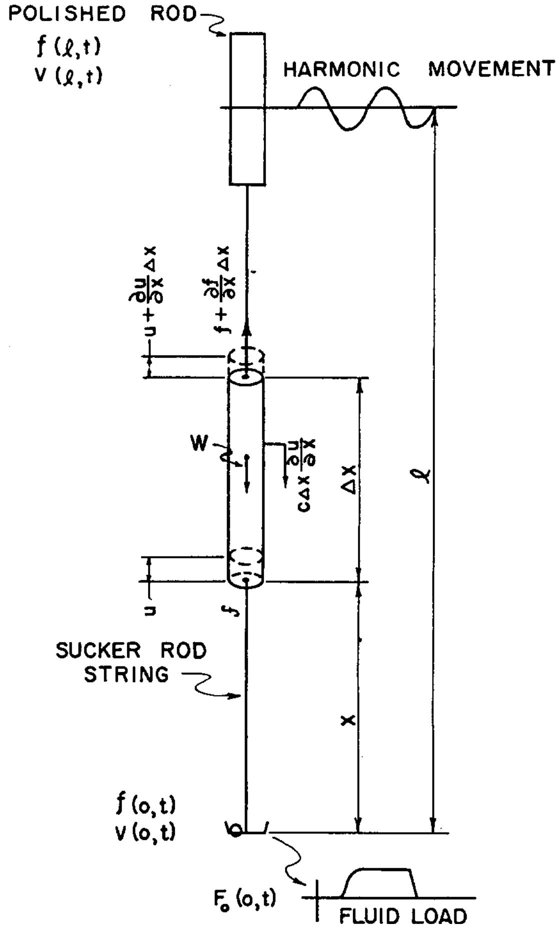


Fig 1

the rod system at any position and at any time.

The variable x or the space coordinate is considered as a constant, and the time variable t as a variable; then the Laplace Transformation of Equations 1 and 2 can be written as

$$f_x\ (x,s) = zv\ (x,s) \qquad 3.$$

$$v_x\ (x,s) = yf\ (x,s) \qquad 4.$$

where

$$z = c + Ms \qquad a.$$

$$y = 1/ks \qquad b.$$

and f_x, v_x denote differentiation with respect to the independent variable x, and (x,s) indicates f or v is a function of both variables x and s.

If Equations 3 and 4 are differentiated with respect to x, two second order ordinary differential equations are obtained as follows:

$$f_{xx}\ (x,s) = zy\ f(x,s) \qquad 5.$$

$$v\ \ (x,s) = zy\ v(x,s) \qquad 6.$$

These equations have solutions of the form

$$f(x,s) = Ae^{\gamma x} + Be^{-\gamma x} \qquad 7.$$

$$v(x,s) = \frac{1}{Z_c}\ )Ae^{\gamma x} - Be^{-\gamma x}) \qquad 8.$$

where

$$\gamma = \sqrt{zy} \qquad c.$$

$$Z_c = \sqrt{z/y} \qquad d.$$

For normal operating conditions, expressions c and d may be approximated by

$$\gamma = \alpha + \beta s \left[\frac{c}{2\ Mfe}\right] + \sqrt{\frac{M}{k}}\ s \qquad e.$$

$$Z_c = D + Es \quad Mk + \frac{c}{2}\sqrt{\frac{k}{M}} \quad \frac{1}{s} \qquad f.$$

In these frequency domain solutions the constants A, B, Z_c and γ are functions of the parameter s; therefore in the real time domain they will be functions of the original parameter t. The constants A and B depend upon the initial boundary conditions while Z_c and γ depend upon the physical parameters of the system. If one designates the boundary conditions symbolically, a solution in terms of these symbols can be obtained. In the final result then it will be necessary to develop suitable expressions for these designated end conditions.

The generalized velocity function in the real time domain is designated by V (ℓ,t) at the polished rod, and the load function is designated by F (o,t) at the pump. This amounts to specifying the nature of the input velocity at the polished rod and the nature of the fluid load on the pump. These time domain functions can be transformed to the frequency domain and substituted into equations 7 and 8 to obtain

$$f(\ell,s) = 2e^{-\gamma\ell} f(o,s)\left(\sum_{n=o}^{\infty} (-1)^n e^{-2n\gamma\ell}\right) + Z_c v(\ell,s)$$

$$2\left(\sum_{n=o}^{\infty} (-1)^n e^{-2n\gamma\ell} - 1\right) \qquad 9$$

$$v(\ell,s) = v(\ell,s)\left(2e^{-\gamma\ell}\sum_{n=o}^{\infty} (-1)^n e^{-2n\gamma\ell}\right) -$$

$$Z_c f(o,s)\left(2\left(\sum_{n=o}^{\infty} (-1)^n e^{-2n\gamma\ell} - 1\right)\right) \qquad 10.$$

where x has been replaced by ℓ and by zero, resulting in expressions for the load at the polished rod and the velocity at the pump.

GENERAL STEADY STATE SOLUTIONS

Inversion of Equations 9 and 10 from the frequency domain to the time domain by formal methods would result in a complete solution containing both transient and steady state terms. Since, in equation 9 above the fluid load at the pump f(o,s) has not been determined, formal inversion would require that this load be expressed analytically throughout the pumping cycle. The exact form of this load during the starting transient is unknown. Therefore, while the transient accompanying the original starting of the pumping operation would be informative and interesting, this load in the presence of damping cannot last longer than a few complete pumping cycles; consequently, it will not be investigated in this paper.

Formal inversion is further complicated by the irrational form of the frequency domain variable s. This function given by expressions c and d can, however, be expanded by the binomial theorem giving the approximations of expressions e and f. It should be noted that the approximations enter only in the inversion of the terms containing the fluid load on the pump and not into the inertia load that is due to the motion of the rod string.

To find the steady state solution (i.e., the time domain solution containing the transient loads reintroduced each cycle by the valves, but excluding the initial starting transient) of the non-harmonic force at the pump, it is first necessary to assume that the time variable load approaches the periodic function, e.g., $F(o,t) = F(o,t-\tau)$ where τ is the period. Observation of dynamometer cards, bottom hole cards and analytical investigations reveals that the load traces reproduce themselves; this reproduction justifies the assumption of the fluid load being periodic. The period is further found to be of the same frequency as is the input velocity, and again justifies the assumption that the fluid load is predominantly due to the static load of the fluid, since the dynamic fluid load will have a natural period dependent upon the properties of the fluid column.

The steady state solution of Equation 9 can be obtained for each term independent of the other term. The first term on the right side of Equation 9 it will be noted contains $f(o,s)$, i.e., the instantaneous fluid load on the pump for any value of the parameter s. One wants to find the value of this fluid load as it traverses the rod string where it will be attenuated and shifted in phase. The steady state solution of this term will then result in an expression of the effect of the lumped fluid load acting on the plunger, as seen at the polished rod. Designating this load by $F_o(\ell,s)$ and performing the inversion to real time space result in

$$F_o(\ell s) = f(\ell,t) - 2\left(\sum_{n=o}^{\infty} (-1)^n F(o,t-(2n+1)\beta\ell)\right) e^{-(2n\ 1)\beta\ell} \qquad 11.$$

where each term on the right-hand side is a periodic function extending from $-\infty \lesssim t \lesssim +\infty$. The result given by Equation 11 can be expressed in better form by recalling that $F_o(\ell,t)$ was restricted to a periodic function with the period $\tau = \frac{2\pi}{\omega}$; therefore, to any desired degree of accuracy, some finite multiple p, of $2\beta\ell$ is equal to some multiple q, of the period, or

$$p(2\beta\ell) = q \quad = q\,\frac{2\pi}{\omega}. \qquad g.$$

This p may be found by taking increasing multiples of $2\beta\ell$ until the result is as close to a multiple of q as desired. It has been found that a practical representation of the rod line effect on the load is given by considering multiples of 1/32 of a wavelength. Since $\frac{\omega\beta\ell}{2\pi}$ is the fractional wavelength, this factor is replaced by a/32 where a is an integer which gives the closest approximation in

$$\frac{a}{32} \approx \frac{\omega\beta\ell}{2\pi} \qquad h.$$

This expression g may be replaced by

$$p = q\frac{1}{2}\ \left(\frac{32}{a}\right) \qquad i.$$

where q is the smallest integer which allows the right side of i to be an integer and this determines p. Note that p is always even.

Examination of Equation 11 shows that there exists an infinite number of sub-series. If these sub-series are collected together according to the way by which the term is shifted along the time axis, and if the typical sub-series is determined, one has a final expression for the steady state fluid load as seen at the polished rod as follows

$$F_o(\ell,t) = \frac{e^{p\alpha\ell}}{\operatorname{Sinh} n\alpha\ell} \sum_{1=o}^{p-1} (-1)^i\, F\left[o,(2i+1)\beta\ell\right] e^{-(2i+1)\alpha\ell} \qquad 12.$$

Equation 12, then, is the final time domain expression for the steady state solution of the first term of Equation 9. It will be noted that to evaluate Equation 12 requires that $F(o,t)$ be determined, i.e., the nature of the fluid load at the coordinate position zero or the pump. The time variable form of this load is assumed and given in Appendix A, where $F(o,t)$ is found to be

$$F(o,t) = f_s \left\{\left(\frac{1}{2} - \operatorname{Cos}\frac{\pi}{\lambda}t\right) S_{(t)} + \frac{1}{2}\left(1 - \operatorname{Cos}\frac{\pi(t-\lambda)}{\lambda}\right) S_{(t-c)} + \left(\frac{t-\phi}{\phi-\theta}\right) S_{(t-\phi)} + \left(\frac{t-\theta}{\phi-\theta}\right) S_{(t-\theta)}\right\} S.F \qquad 13.$$

The second term of Equation 9 can be stated as

$$F_r(\ell s) = Z_c v(\ell s) \tanh \gamma\ell \qquad 14.$$

where $F_r(\ell,s)$ is the dynamic inertial load at the polished rod arising from the sinusoidal manipulation of the rod string neglecting the fluid load. In the steady state Equation 14 becomes

$$F_r(\ell, i\omega) = \sqrt{Mk - i\frac{ck}{\omega}}\,.\,v(\ell, i\ )\tan\sqrt{i\frac{c\omega}{k} - \omega^2\frac{M}{k}} \qquad 15.$$

which is of the general form

$$F_r(\ell, i\omega) = A_\omega\ \underline{|\phi_\omega}\ v(\ell, i\omega)\, B_\omega\ \underline{|\psi_\omega}\,. \qquad 16.$$

Since the fluid load has been constructed and represented as a periodic load and the rod line system is linear, the non-linearites have disappeared and the principle of superposition applies. Therefore, the effect of the various input velocity harmonics acting on the distributed rod string may be determined separately and added directly. The input or driving function velocity expression for the usual harmonics is developed in Appendix B, and can be expressed as

$$v(\ell,s) = V\operatorname{Sin}\omega t + \lambda_2 V\operatorname{Sin}(2\omega t + \theta_2) + \lambda_3 V\operatorname{Sin}(3\omega t + \theta_3). \qquad 17.$$

Then the second term of equation 9, the inertia load considering distorted sinusoidal input, can be expressed in the time domain as

$$F_r(\ell,t) = A_1B_1V\operatorname{Sin}(\omega t + \phi_1 + \psi_1) + \lambda_2A_2B_2V\operatorname{Sin}(2\omega t + \theta_1 + \phi_2 + \psi_2) + \lambda_2A_3B_3V\operatorname{Sin}(3\omega t + \theta_2 + \phi_3 + \psi_3)\,. \qquad 18.$$

Evidently then, by the method superposition, the results of Equation 12, giving the effect of the fluid load, and the results of Equation 18, giving the effect of the rod inertia, are directly additive and result in the final expression for the polished rod load (PRL).

$$\begin{aligned} PRL &= F_o(\ell,T) + F_r(\ell,t) \\ &= \frac{e^{p\alpha\ell}}{\operatorname{Sinh} p\alpha\ell} \sum_{1=0}^{p-1} (-1)^i\, F\left(o,(2i+1)\beta\ell\right) e^{-(2i+1)\alpha\ell} \\ &\quad + A_1B_1 V\operatorname{Sin}(\omega t + \phi_1 + \psi_1) \\ &\quad + \lambda_1 A_2B_2\ V\sin(2\omega t + \Theta_1 + \phi_2 + \psi_2) + \\ &\quad \lambda_2 A_3 B_3 V\sin(3\omega t + \Theta_2 + \phi_3 + \psi_3) \end{aligned}$$

Each of the above factors is defined in terms of well parameters in Appendix C. It is this expression that should be programmed for computer calculation of the polished rod load as time or crank angle ωt varies.

ADDITIONAL OBSERVATIONS

It will be recalled that dynamometer cards, as recorded at the well, are load-displacement traces. The displacement of the polished rod may be expressed for the fundamental of a sinusoidal input as

$$x = x_m \text{Cos } \omega t. \qquad \text{j.}$$

In the absence of the fluid load at the pump, the inertia load at the polished rod can be expressed as

$$f = f_m \text{ Sin}(\omega t + p) \qquad \text{k.}$$

Eliminating t between these expressions gives an implicit relationship for the force as a function of the displacement. The resulting analytical expression is the equation of a family of ellipses. For the fundamental, the following typical polished rod diagrams result from the inertia load depending upon the values of the phase angle p.

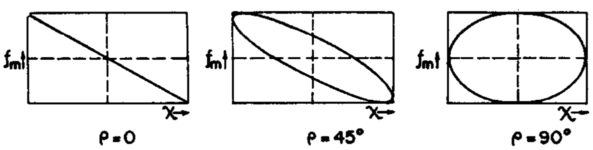


Fig. 2

The second harmonic in the input velocity would result in a force of the nature shown below for various phase angles:

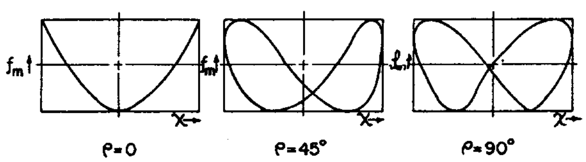


Fig. 3

The above figures are easily recognizable as the familiar Lissajou patterns obtained for frequency-phase relationships in distributed systems. If one considers, then, only the distributed rod line mass (the fluid is ignored), a system containing approximately 10 per cent second harmonic and for a 30° phase angle has a typical basic card shape as shown below:

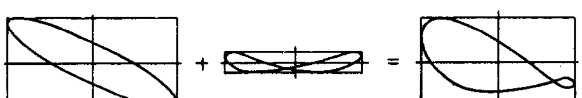


Fig. 4

The typical characteristics of many dynamometer cards are evident from this composite representation and if one allows for the added fluid load and additional coulomb friction in the well system, many cards can be visualized. The nature of the fluid load is discussed further in the following sections on graphical construction and the Lowery well example.

GRAPHICAL CONSTRUCTION

An acceptable approximation of a dynamometer card shape may be found graphically by determining or assuming the nature of the fluid load on the pump as a function of time, and adding to this the inertia load caused by movements of the rod line. To obtain the fluid load, a card shape is assumed as in Figure 5, where the magnitude f_s represents the static weight of the fluid on the pump This load is then translated to the right by an amount $\beta \ell + \delta$. The negative of the fluid load is shifted to the right $3\beta\ell + \delta$ and attenuated by $e^{-\alpha \ell}$ as in Figure 6. This process is repeated until the shifted load concides with the original load, which results in the composite fluid load as seen at the polished rod (Figure 6).

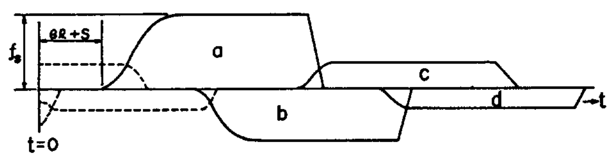


Fig. 5

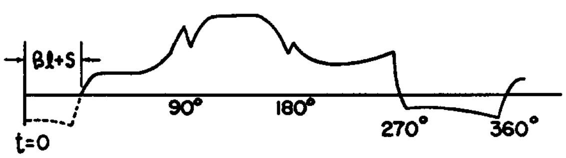


Fig. 6

It is this time referenced load that must be added to the elliptical load caused by rod moment. And direct addition of the fluid load from Figure 6 to the inertia load of Figure 4 results in the following graphical construction for the polished rod load:

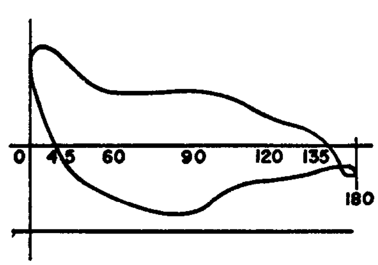


Fig. 7

Example: The above general graphical procedure will be applied to the Lowery well for which published data has been made available.

The well parameters are as follows:

$\omega = 1.67$ radians/ sec

$x_m = .9125$ ft

$V_m = 1.52$ ft/sec

$F_m = 1450$ lb

$Z_c \approx D \approx 820$

$\beta = .233$ sec

$a = .16$

$\lambda_1 = \lambda_2 \; 0$

$\tau = \frac{2\pi}{\omega} = 3.26$ sec

$\lambda = f/10 = .326$ sec

$\phi = \frac{17}{30}\tau = 2.13$ sec

$\Theta = \frac{44}{75}\tau = 2.2$ sec

$\delta \cong \beta = .233$ sec

$p = 8$

$q = 1$

$a = 2$

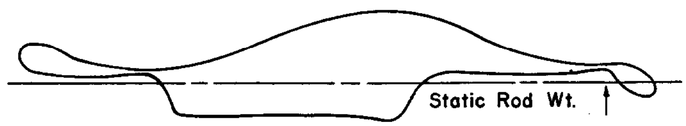
Then
$$F(\ell,t) = 510 \ \mathrm{Cos}\,\omega t \ + 2.16 \sum_{1=o}^{7} (-1)^n F\left(o,t-2.33 \,(2i-1)\, e^{}\,(2i+1)\right) e^{-.16(2i+1)}$$

where

$$F(o,t) = 1450 \left\{ \frac{1}{2}(1 - \mathrm{Cos}\ 9.65t)(S_o + S_{(t-.326)}) + \left(\frac{t-2.2}{.07}\right) S_{(t-.213)} + \left(\frac{t-2.2}{.07}\right) S_{(t-2.2)} \right\} \mathrm{S.F.}$$

and S. F. shifts the expression to the right in time by $\delta \approx \beta \ell = .233$ sec and enables F(o,t) to repeat itself every $\frac{2\pi}{1.67}$ sec.

The above expression, when plotted with t as the variable, results in the following comparison of dynamometer card and measured card:



Graphical and Calculated

Measured

Fig. 8

CONCLUSIONS

It should be noted that the unit geometry in general (along with imperfect though adequate counterbalance) introduces distortion containing small percentages of second, third, and fifth harmonics. Of special interest is the fact that application of the final analysis to pertinent well situations will reveal that, many times, the intermittent fluid load is reflected at the polished rod in such phase relationship as to appear as additional second harmonic distortion. Therefore, this load phase relationship, together with unit distortion, makes the analysis of the second harmonic critically important since normal pumping speeds often result in second harmonic resonance. Resonance at the second harmonic frequency is accompanied by major changes in the load time plot for only minor variations in the input velocity or pumping speed. And it should further be noted that a system operating below resonance may appear as an inertia or equivalent electrical inductive system; while passing through resonance to frequencies somewhat higher, the same system may appear as an elastic or an equivalent electrical capacitive system. This complete change in system nature may account for the erratic change in card shape with little change in pumping speeds.

Determination of the various components in their proper time phase relationships which characterize dynamometer cards has been accomplished. But the primary disadvantage in the application of the procedure is the questionable shape of the fluid load.

Additional research is needed in the area of the bottom hole load function, particularly as it may be modified by higher harmonics of the driving velocity at the pump.

It would also be of interest to find the bottom hole velocity and displacement. This process should be straightforward; however, the final step of correlating the fluid load with polished rod cards may prove difficult since accurate timing or phasing data are not available.

Numerical values of the damping constant c for various well conditions also need to be examined more closely. This examination would become especially important in trying to duplicate analytically the well conditions at resonance for the various harmonics since small deviations in c cause large deviations in the theoretical polished rod card.

APPENDIX A

FLUID LOAD AT THE PUMP

It is now necessary to assume the time form of the lumped fluid load at the pump. From observations of recorded time dependent pump load diagrams and from theoretical considerations, the general form is known. Figure A1 shows the load time variation, from which an analytical expression in terms of the magnitude (static fluid load f_s), the rate of load increase, time duration, and the rate of decrease are given by Equation AI.

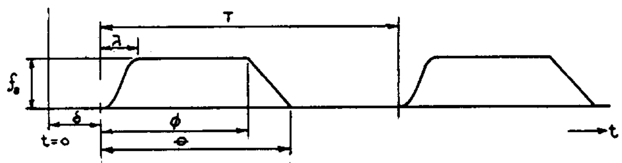


Fig. A1

$$f(o,s) = \frac{f_s e^{\delta s}}{1 - e^{-\tau s}} \; \frac{1}{2}\left(\frac{1}{s} - \frac{s}{s^2 + (\pi/\lambda)^2}\right)(1 + e^{-\lambda s}) + \frac{e^{-\phi s} - e^{-\Theta s}}{s^2 (\phi - \Theta)}. \qquad \text{AI}$$

The variables λ, ϕ, Θ, and δ have not been fully determined; however, nominal values for properly functioning equipment appear to be

$$\lambda \approx \tau/10 \approx \frac{1}{10\ \text{SPM}}$$

$$\phi \quad 17/30 \approx \frac{17}{30\ \text{SPM}}$$

$$\Theta \quad 44/75 \approx \frac{44}{75\ \text{SPM}}$$

$$\delta \approx \beta = \sqrt{\frac{M}{k}}$$

To represent the fluid load in the time domain as required by Equation 12 demands inversion of the frequency domain expression giving

$$F(o,t) = f_s \left\{ \frac{1}{2}(1 - \text{Cos}\frac{\pi}{\lambda}t)\, S_{(t)} + \frac{1}{2}(1 - \text{Cos}\frac{\pi(t-\lambda)}{\lambda})\, S_{(t-c)} + \left(\frac{t-\phi}{\phi-\Theta}\right) S_{(t-\phi)} + \left(\frac{t-\Theta}{\phi-\Theta}\right) S_{(t-\Theta)} \right\} \text{S.F.} \qquad \text{AII}$$

where S.F. is a factor which shifts the whole expression to the right in time by $\delta = \beta \ell$ seconds and also causes the expression to repeat periodically every seconds.

APPENDIX B

INPUT OR DRIVING FUNCTION VELOCITY

In general, the input velocity is not a perfect sine wave. Unit geometry, counterbalance, power available, etc., result in a distorted input generally containing significant second and third harmonics. These harmonics often materially alter the final card shape, because, for example, a system driven at one-half resonance of the natural fundamental frequency of the rod string will be operating at resonance for the second harmonic. Since the phase relationship of the forces arising from the second harmonic and since the fluid load at the pump can add in phase, the second harmonic distortion can result in forces or loads which determine the predominant shape of the dynamometer card. The frequency domain expression for the input velocity required in Equation 9 -- with one taking into account the distorted input -- can be expressed as

$$v(\ell,s) = V\text{Sin}\,\omega t + \lambda_2 V\text{Sin}(2\omega t + \Theta_2) + \lambda_3 V\text{Sin}(3\omega t + \Theta_3).$$

In this expression λ_2 and λ_3 are the fractional harmonic content of the second and third harmonics, and Θ_2 and Θ_3 are the phase angles of the respective harmonics.

APPENDIX C

The various terms in Equation 19 are defined below in terms of well parameters:

$$A_1 = |a_1 + ia_1'| = \left|\sqrt{Mk - 1\frac{ck}{\omega}}\right|$$

$$\phi_1 = \tan^{-1}\frac{a_1'}{a_1}$$

$$B_1 = |b_1 + ib_1'| = \left|\tanh\sqrt{\frac{1\omega c}{k} - \frac{M\omega^2}{k}}\right|$$

$$\psi_1 = \tan^{-1}\frac{b_1'}{b_1}$$

$$A_2 = |a_2 + ia_2'| = \left|\sqrt{Mk - i\frac{2k}{2\omega}}\right|$$

$$\phi_2 = \tan^{-1}\frac{a_2'}{a_2}$$

$$B_2 = |b_2 + ib_2'| = \left|\tanh\sqrt{i\frac{2\omega c}{k} - \frac{M(2\omega)^2}{k}}\right|$$

$$\psi_2 = \tan^{-1}\frac{b_2'}{b_2}$$

$$A_3 = |a_3 + ia_3'| = \left|\sqrt{Mi - i\frac{ck}{3\omega}}\right|$$

$$\phi = \tan^{-1}\frac{a_3'}{a_3}$$

$$B_3 = |b_3 + ib_3'| = \left|\tanh\sqrt{i\frac{3\omega c}{k} - \frac{M(3\omega)^2}{k}}\right|$$

$$\psi_3 = \tan^{-1}\frac{b_3'}{b_3}$$

ω = the fundamental input velocity frequency

M = mass of the rod string

k = spring constant of the polished rod

c = damping along polished rod (3)

REFERENCES

(1) J. R. Norton. "Dynamic Loads in Sucker Rods," Drilling and Production Practice, American Petroleum Institute, 1960.

(2) J. R. Norton, "New Method Predicts Rod Input Power Requirements," Petroleum Engineer, September 1960.

(3) "Counterbalance Can Be Determined from Dynamic Unbalance and Friction Loads," J. R. Norton, Petroleum Engineer, February 1961.

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